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Brief Notes

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Zener's Crack and the *M*-Integral

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In a pair of bonded solids, the interface may block dislocation gliding. The pileup may cause a crack to nucleate either on the interface, or in one of the solids. The model, proposed by Zener half a century ago, has been analyzed in various forms. This note shows that the energy release rate of the crack can be calculated by an application of the M-integral. Both solids are anisotropic, and the interface is flat. The result leads to a discussion of the crack orientation. [S0021-8936(00)00701-7]

Introduction

Zener [1] proposed that a dislocation pileup concentrates stress, which may cause a crack to nucleate. Any crystallographic discontinuity (e.g., a grain boundary or a phase boundary) may act as an obstacle to block dislocation gliding. In a pair of bonded solids, when the interface blocks dislocations, the crack can nucleate either on the interface (Fig. 1(*a*)), or in one of the solids (Fig. 1(*b*, *c*)). The model has been analyzed by many authors in the last 50 years, as reviewed by Cottrell [2] and more recently by Cherepanov [3] and Fan [4]. A main result is the energy release rate of the crack, which has been obtained by solving elasticity boundary value problems for various special cases. When the two solids have dissimilar elastic properties, only the case where the crack lies on the interface (Fig. 1(*a*)) has been solved. This note calculates the energy release rate by using a path-independent integral, following the procedure of Freund [5]. He gave several examples of cracks in an isotropic and homogeneous solid. Here the two solids are anisotropic and dissimilar. The crack can be either on the interface, or in one of the solids.

Calculation

Consider an elastic solid in a state of plane-strain deformation. Knowles and Sternberg [6] introduced the following integral:

$$M = \int_C (w n_\alpha - t_\beta u_{\beta,\alpha}) x_\alpha dS. \tag{1}$$

The integral is over a closed curve C in the plane. Here x_α is the rectangular coordinate, n_α the unit normal vector to C, w the strain energy density, u_α the displacement, and t_α the traction. If the solid is homogeneous along rays from the coordinate origin and if C encloses no singularity, $M=0$; that is, M is a path-independent integral. For a pair of dissimilar solids bonded on a flat interface, placing the coordinate origin at any point on the interface satisfies the homogeneity requirement.

If C does enclose a singularity, M may not vanish. For example, Rice [7] showed that the M-integral over a closed curve around a dislocation equals the pre-logarithmic factor of the dislocation energy. Consider a dislocation of the Burgers vector $\boldsymbol{b}$ lying on the

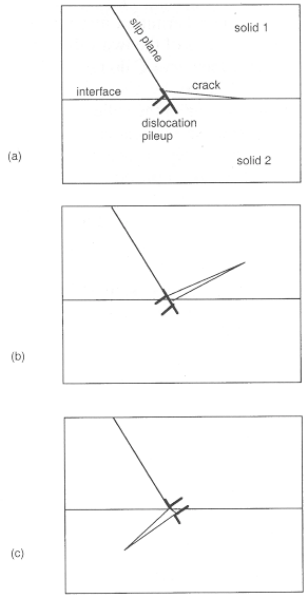

Fig. 1 Zener's model. Dislocations pile up at an obstacle, i.e., the interface between two solids. The intense stress causes a crack to nucleate either on the interface, or in one of the solids.

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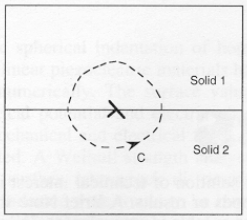

Fig. 2 A dislocation lies on an interface. An arbitrary curve *C* encloses the dislocation.

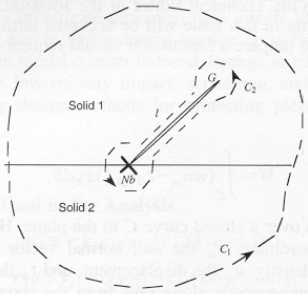

Fig. 3 A superdislocation on the interface, and a crack either on the interface or in one of the solids. The *M*-integral is evaluated over the two closed curves.

interface between a pair of solids (Fig. 2). Place the coordinate origin at the dislocation and a curve C around the dislocation. The M-integral is

$$M=\frac{1}{2\pi}\boldsymbol{b}^T\boldsymbol{H}^{-1}\boldsymbol{b}. \tag{2}$$

Here $\boldsymbol{H}$ is a positive-definite hermitian matrix, and can be calculated once the elastic constants of the two solids are given [8]. The location and the shape of the curve C do not affect the value of M. Furthermore, M is unaffected by the presence of an external boundary or other singularities in the solids, so long as C encloses no other singularities than the dislocation. This is understood by making C tightly surround the dislocation, where the singular stress field due to the dislocation prevails over the stress field due to other sources.

It has been shown that, for a pair of anisotropic solids with a fixed relative orientation, under an in-plane coordinate rotation, $\boldsymbol{H}$ transforms like a second-order tensor [9–11]. Consequently, if the interface rotates with the coordinate, but the Burgers vector $\boldsymbol{b}$ remains fixed relative to the solids, the number $\boldsymbol{b}^T\boldsymbol{H}^{-1}\boldsymbol{b}$ is invariant, and so is the value of M.

Figure 3 illustrates a flat interface between a pair of semi-infinite solids. The interface blocks N dislocations, each of the Burgers vector $\boldsymbol{b}$. A crack of length l lies either on the interface or in one of the solids. All the dislocations glide into the crack and blunt one crack tip. The other crack tip advances either on the interface or in one of the solids. We now calculate the energy release rate G at the advancing crack tip. Place the origin of the coordinates at the point where the dislocations are blocked. Because of the path-independence, the M-integral over any curve enclosing the dislocation-crack complex has the same value. First look at the curve C_1 far away from the complex. At a distance far from the dislocations, $r\to\infty$, all the dislocations behave collectively like a single superdislocation having the Burgers vector $N\boldsymbol{b}$; the stress field due to the superdislocation decays as $1/r$, but the modification due to the presence of the crack decays as $1/r^2$. Consequently, the M-integral over C_1 is the same as that over an isolated interface dislocation of the Burgers vector $N\boldsymbol{b}$. Equation (2) is applicable once $\boldsymbol{b}$ is replaced by $N\boldsymbol{b}$. Next look at the curve C_2 tightly surrounding the dislocation-crack complex. The stress field near the origin is now less singular than $1/r$, so that the small circle around the origin does not contribute to the M-integral. Nor do the traction-free crack faces. It can be shown that the M-integral over the small circle around the crack tip away from the origin equals lG [5]. Equating the M-integrals evaluated over C_1 and C_2, one obtains that

$$G=\frac{N^2}{2\pi l}\boldsymbol{b}^T\boldsymbol{H}^{-1}\boldsymbol{b}. \tag{3}$$

This is the desired result.

Discussion

For the case where the crack lies on the interface, Fan [4] solved the elasticity boundary value problem using a complex variable method, and calculated the energy release rate from the stress field. Equation (3) agrees with his result. Furthermore, this note demonstrates that the same equation is applicable to a crack inside one of the solids.

Let γ_1, γ_2, and γ_i be the surface energy per unit area of solid 1, solid 2, and the interface, respectively. When a crack advances a unit area, the surface energy increases by Γ, where $\Gamma=2\gamma_1$ if the crack is in solid 1, or $\Gamma=\gamma_1+\gamma_2-\gamma_i$ if the crack is on the interface. Crack can nucleate if the energy release rate compensates the surface energy increase, $G=\Gamma$, namely,

$$\frac{N^2}{2\pi l}\boldsymbol{b}^T\boldsymbol{H}^{-1}\boldsymbol{b}=\Gamma. \tag{4}$$

The crack length l scales with N^2, everything else being fixed. It is also interesting to predict crack orientation from this model. The value of Γ depends on the crystalline orientation. As pointed out above, once the relative orientations of the two solids and the slip plane are fixed, the factor $\boldsymbol{b}^T\boldsymbol{H}^{-1}\boldsymbol{b}$ is invariant with the rotation of the interface or the crack. Consequently, when the relative orientations of the two solids and the slip plane are fixed, the crack orientation is entirely selected by the anisotropy in Γ. According to this model, anisotropy of elastic constants plays no role in selecting the crack orientation.

Acknowledgments

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