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Flaw sensitivity of stochastic elastic materials

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Abstract
A material-specific length, called the flaw sensitivity length or fractocohesive length, is determined by measuring the strength of samples that contain cracks of various lengths. When the crack length is small compared with the fractocohesive length, the strength is unaffected by the crack. When the crack length is large compared with the fractocohesive length, the strength reduces as the crack length increases. Here we study how the fractocohesive length is affected by the stochastics of the constituents of a material. We simulate a model system, a truss in which the constituents are linearly elastic members forming a geometrically periodic lattice. The stochastics are represented by the scatter of strength among the members. The fractocohesive length scales with the length of each individual member, but the prefactor increases with the degree of scatter in member strength. The fractocohesive length can be much larger than the constituents of a material when the constituents have pronounced statistical variation.

Keywords
Fracture mechanics, flaw sensitivity, stochastic truss, fractocohesive length, stochastic elasticity

1. Introduction

Griffith [1] showed that the strength of a material is reduced by crack-like flaws. Subsequent studies have shown that the strength of a material is unaffected by crack-like flaws if the flaws are small compared with a material-specific length, r_F (Figure 1).

This material-specific length is called the flaw sensitivity length, or the fractocohesive length. The values of fractocohesive length are known for many materials [2]. For example, in silica, the deformation is linearly elastic except for atoms immediately around the crack tip, and the fractocohesive length of silica is ~ 1 nm. That is, the fractocohesive length of silica is on the order of the size of the constituents of the material, i.e., an individual covalent bond. Because crack-like flaws above 1 nm are inevitable in samples

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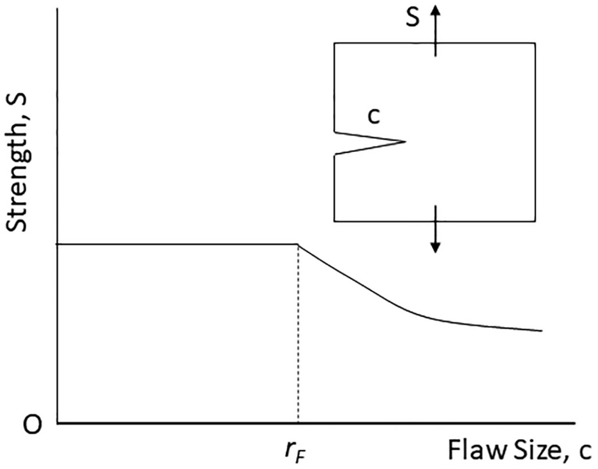


Figure 1. Strength as a function of length of crack-like flaw, c. A material-specific length r_F exists. When $c<r_F$, the strength is insensitive to the flaw size. When $c > r_F$, the strength decreases as the flaw size increases.

of silica of macroscopic dimension, the measured strength of the macroscopic samples is sensitive to flaws. Furthermore, because the sizes of the flaws are stochastic, the measured strength of silica has a large statistical variation from sample to sample [3].

By contrast, in a metal or a plastic, crack growth is accompanied by plastic deformation in a zone around the crack tip [4]. The plastic deformation greatly reduces stress concentration at the crack tip. The size of the plastic zone is often much larger than the atomic dimension. Consequently, the fractocohesive lengths of metals and plastics are much larger than the atomic dimension. For example, consider a representative material of a fractocohesive length of 1 mm. For a well-prepared sample, crack-like flaws are typically smaller than 0.1 mm. Consequently, such a material is flaw-insensitive.

Large fractocohesive lengths have also been observed for materials for which negligible inelastic deformation accompanies crack growth. For example, for a polyacrylamide hydrogel, the stress–strain curve shows negligible hysteresis, because water has low viscosity, and the polymer chains in the polyacrylamide hydrogel have low frictional interactions [5]. However, the polyacrylamide hydrogel has a fractocohesive length of about 1 mm. This fractocohesive length is much larger than the size of the constituents of the hydrogel, namely, the end-to-end distance of an individual polymer chain. The origin of the fractocohesive length in such a highly elastic material is obscure. It has been suggested that the large difference between the length of constituents and the fractocohesive length results from statistical variation in lengths among the chains in the polymer network. Without statistical variation in chain length, damage will be concentrated at a crack tip. With statistical variation, damage will be diffused. This picture, however, has not been studied theoretically.

When a networked material is stretched, members break one by one progressively leading to overall fracture. The breaking of a member damages the networked material but does not fail the material. The fracture behavior of a networked material depends on the individual members in the network, how the members are connected, as well as how the members interact with one another. In silica, silicon and oxygen atoms form an amorphous network of siloxane bonds, and the mesh size of the network is the atomic spacing. In polyacrylamide hydrogel, monomers link into polymer chains, and polymer chains crosslink into a network. The polymer chains not only interact with one another through crosslinks but also through entanglements and water molecules.

Here we use numerical simulation to study the stochastic origin of the fractocohesive length in a relatively simple class of networked materials: trusses. A truss is a network of solid rods, which interact through joints only. As an idealized model, we consider a triangular lattice (Figure 2). Each member of the truss is linearly elastic, characterized by modulus and strength. We assume that the modulus is the same for all members, but the strength varies from member to member stochastically. This model sets up a competition between stress concentration and statistical distribution. The stress concentration acts around a crack tip. The statistical distribution in member strength acts everywhere in the sample.

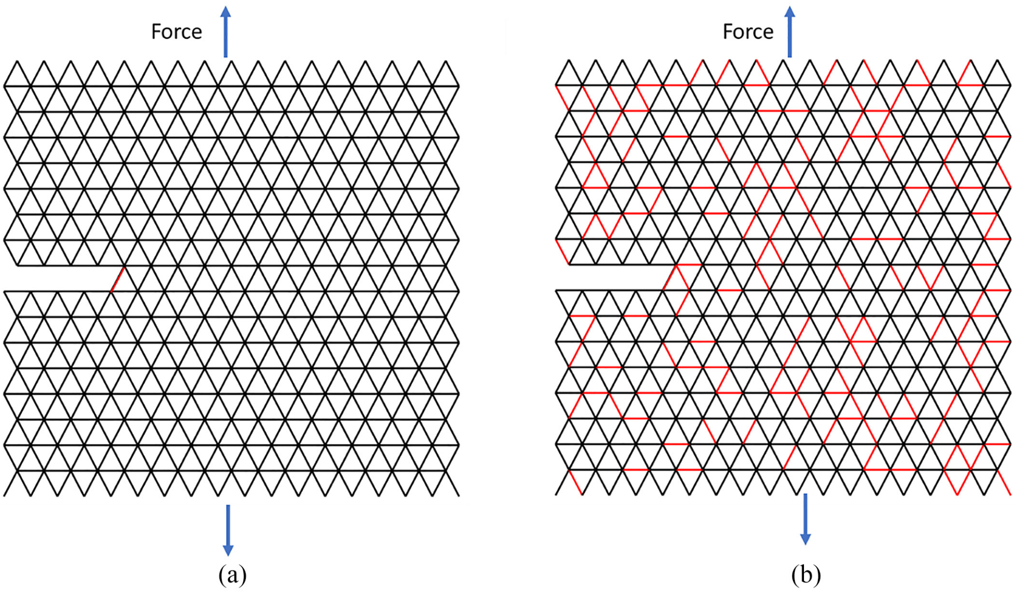


Figure 2. Stochastic constituents reduce flaw sensitivity. In an elastic lattice, a crack concentrates stress. (a) If all the members have the same strength, the sample fractures by breaking the member at the crack tip. (b) If the members have large statistical variation in strength, the sample fractures by distributed damage.

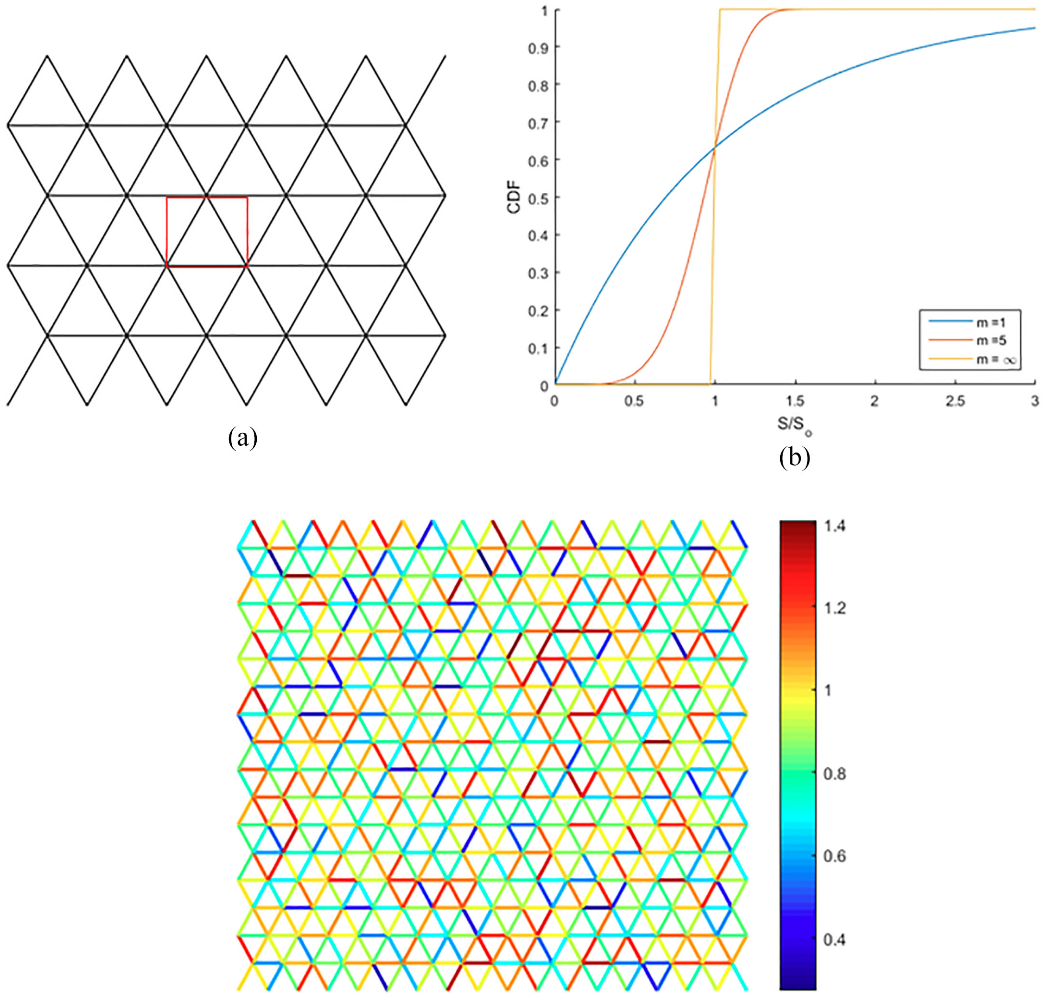
Consequently, the weakest member is less likely to be at the crack tip. For a truss in which the statistical variation in member strength is narrow, stress concentration prevails, so that the crack grows by breaking the member at the crack tip, and the fractocohesive length is comparable with the individual member length (Figure 2(a)). For a truss in which the statistical variation in member strength is wide, statistical distribution prevails, so that, before the member at the crack tip breaks, some members away from the crack tip break, and the fractocohesive length can be much larger than the individual member length (Figure 2(b)). That is, stochastics among members reduces the flaw sensitivity of a material.

We represent the statistical variation in member strength by the Weibull distribution and use the Weibull shape parameter to quantify the degree of statistical variation. For each truss, with or without a crack, we simulate stress–strain curves using an algorithm in which members are progressively eliminated as they break. The peak stress of the stress–strain curve identifies the truss strength. By plotting the truss strength as a function of crack length, we identify the fractocohesive length. For a truss in which all members have the same strength, stress concentration at the crack tip prevails, and the crack greatly knocks down the truss strength. For a truss in which the member strength scatters widely, the statistical distribution prevails, and the truss strength is insensitive to the crack, so long as the crack is not too large. Our simulations confirm that statistical variation in member strength increases the fractocohesive length.

Fracture of networked materials has emerged as an active field of study. Toughness has been calculated for periodic, deterministic trusses, as well as for trusses in which the lengths of the members are stochastic [6]. Through both experiments and simulations, the fracture toughness of three-dimensional trusses has been studied [7]. Progressive damage of stochastic fishnet trusses, in which no precracks are introduced, has been simulated to study the low-strength tail of the truss failure [8], as well as the effect of specimen size [9]. Simulations of trusses with geometrically nonlinear members which rupture kinetically have been performed [10]. In this paper, we focus on a geometrically periodic truss with stochastic member strengths, in which progressive damage leads to a fractocohesive length much larger than the member length.

2. Method of simulation

Simulations are performed on a sheet of triangular lattice of $N_{u,x}$ unit cells in the x-direction and $N_{u,y}$ unit cells in the y-direction (Figure 3(a)). Each member has a length b, so the truss has a width $L = N_{u,x}b$ and a height $H = \sqrt{3}N_{u,y}b/2$. Each member is a rod with cross-sectional area A. The left ($x = 0$) and



(a)

(b)

(c)

Figure 3. A lattice with stochastic strength of individual members. (a) Periodic triangular lattice. (b) The strength of members follows the Weibull cumulative distribution function. (c) An example of stochastic S/S_0 assigned to the members according to the Weibull distribution with $m = 5$.

right ($x = L$) edges of the truss are constrained with zero displacement in the x-direction. The bottom ($y = 0$) edge of the truss is constrained with zero displacement in the y-direction. The top ($y = H$) edge of the truss is stretched by displacements in the y-direction. Because the left and right edges of the rectangular sample are constrained, these boundary conditions mimic the so-called “pure-shear test” [11]. We simulate the progressive damage of the truss, compute the force as a function of displacement, and identify the truss strength by the peak force of the force–displacement curve. We also create models with cracks of length c and apply the same boundary conditions as described above.

Each individual member is linearly elastic with Young’s modulus E and strength S. We assume that the modulus is the same for all members, but the strength varies from member to member according to the Weibull distribution, with the cumulative distribution function (CDF):

$$CDF = 1 - exp\left(-\left(\frac{S}{S_0}\right)^m\right), \tag{1}$$

where S_0 indicates the scale of the strength, and m is the shape parameter. The Weibull distribution is commonly chosen to represent strength [12]. The shape parameter m characterizes the degree of scatter

(Figure 3(b)). The smaller the value of m, the larger the scatter in member strength. When $m \to 1$, the CDF approaches an exponential distribution, and the members have a broad distribution of strength. When $m \to \infty$, the CDF approaches a step function, and all members have the same strength S_0. For a given truss and a value of m, members in the truss are assigned a random strength following the Weibull distribution (Figure 3(c)). This is done by first randomly generating a number between 0 and 1. The random number is then taken as a value of the CDF in equation (1), which is solved for the assigned strength, S.

Once every member of a truss is assigned a strength, we monotonically increase the displacement of the top boundary and progressively eliminate broken members. A tensile member breaks when the stress in the member reaches its strength, but a compressive member is assumed not to fail (i.e., buckling and crushing are assumed not to occur). We have simulated the progressive damage using the commercial software ABAQUS. However, ABAQUS requires many small increments to converge whenever a member breaks. This makes simulations run exceedingly slow and even stall for a truss having many members and a wide statistical variation in member strength.

Note that the displacement is small, and every member is linearly elastic before it breaks. Consequently, between the breaking of individual members, the truss is linearly elastic. To take advantage of this linearity, we write a program with the following algorithm. In each step, apply an arbitrary displacement, assume no member breaks, and solve the linear elastic truss. For each surviving member, calculate the ratio of its stress to its strength. The member with the highest ratio will break in this step. We use this ratio to linearly scale the applied displacement and the reaction force to the point at which the critical member breaks. Remove the critical member. Repeat this procedure to break the next member. This algorithm is similar to the algorithm used by Luo and Bažant [8].

This algorithm results in a set of points on the force–displacement plane, A, B, C, D, and so on (Figure 4). Each point represents the breaking of a single member. We next consider how these points obtained from the simulation can be related to an experiment. As a schematic illustration, assume that the simulated points are arranged as follows. The applied displacements at points B and C are smaller than the applied displacement at point A, but the applied displacement at point D is larger than the applied displacement at point A. We will only consider an experiment in which the applied displacement is programmed to increase with time. Assume that the applied displacement increases slowly compared with the process of breaking of an individual member. In such an experiment, after member A breaks, the applied displacement will be greater than the displacement at which members B and C will break. Hence, we hold the displacement fixed at the displacement at which member A breaks and allow members B and C to break which causes the reaction force to reduce. The drop in force will stop at point D′, which is determined by linearly scaling point D to the displacement at point A. Every time a member breaks the stiffness of the truss decreases, i.e., the slope of line O–A is less than the slope of line O–D. The force at D can be either lower or higher than the force at A. At point D′, the members corresponding to points A, B, and C have been removed, but the member corresponding to point D remains intact. Point D′ represents the state of equilibrium of the truss. We repeat this procedure to construct the entire force–displacement curve.

This algorithm allows us to run many simulations for an $N_{u,x} = N_{u,y} = 80$ truss. We simulate trusses with crack lengths of $0b$, $2b$, $4b$, $10b$, and $20b$, and of 1.33, 2.1, 3.7, 5.8, 9.6, 19.9, 40.3, and ∞. For each crack length and each finite m value, trusses with five different randomizations were simulated.

3. Strength

Each unit cell has two slanted members inside, and another two horizontal members shared with adjacent unit cells (Figure 1(a)). Thus, the net number of members per unit cell is three, and the volume of solid per unit cell is $3bA$. The unit cell has a width of b, and a height of $\sqrt{3}b/2$. The truss is two dimensional, consisting of a single layer of elements. For scaling purposes, we take the thickness of the truss to be $\sqrt{A}$. Thus, the volume of a unit cell is $\sqrt{3A}b^2/2$. The volume fraction of solid in the truss is [13]:

$$\rho = 2\sqrt{3}\frac{\sqrt{A}}{b}. \tag{2}$$

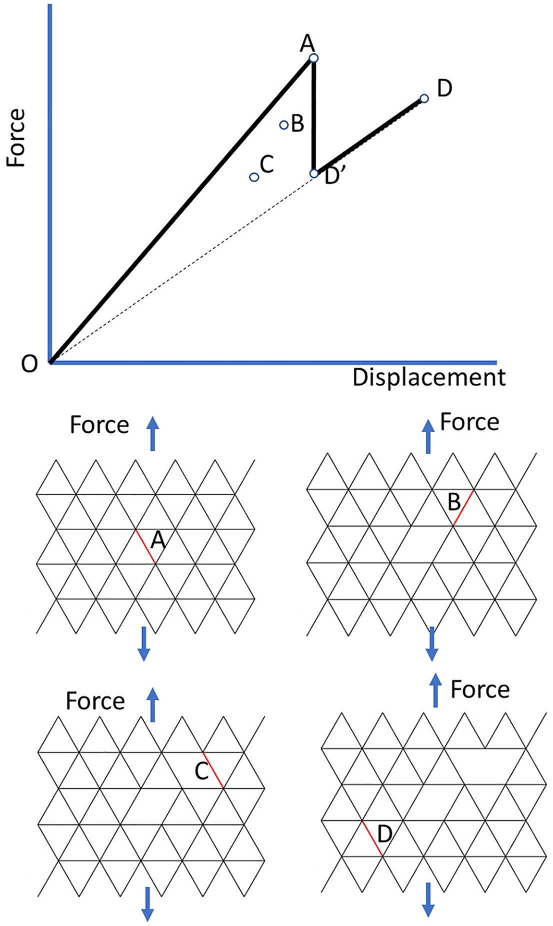


Figure 4. A schematic of the force–displacement curve for a truss with a sequence of broken members.

When displacement δ is applied in the vertical direction of the truss, a reaction force F is developed in the vertical direction. This reaction force acts on the cross-sectional area of the truss, $\sqrt{A}bN_{u,x}$, where $N_{u,x}$ is the number of unit cells in the x-direction. Define the stress by the reaction force divided by the cross-sectional area of the truss:

$$\sigma = F/\sqrt{A}bN_{u,x.} \tag{3}$$

The height of the truss is $\sqrt{3}bN_{u,y}/2$, where $N_{u,y}$ is the number of unit cells in the y-direction. Define the strain of the truss by the displacement δ divided by the height of the truss:

$$\varepsilon = 2\delta/\sqrt{3}bN_{u,y}. \tag{4}$$

To normalize stress and strain, consider an ideal truss with no crack, with every member having the same strength S_0. Each unit cell of the truss has two slanted members that carry the load, but the horizontal members do not contribute to the reaction force (Figure 3(a)). The two slanted members break at the same axial force of AS_0. They are oriented at an angle, so the resultant vertical force from $N_{u,x}$ unit cells is $\sqrt{3}AS_0N_{u,x}$. Consequently, the strength of the ideal truss is $\sqrt{3}AS_0N_{u,x}/\sqrt{A}bN_{u,x} = \sqrt{3A}S_0/b$. Recognizing equation (2), this ideal strength can be written as $\rho S_0/2$. For a truss of stochastic member strength, we normalize the stress by the ideal strength:

$$\frac{\sigma}{\rho S_0/2} = \frac{F}{\sqrt{3}AS_0N_{u,x}}. \tag{5}$$

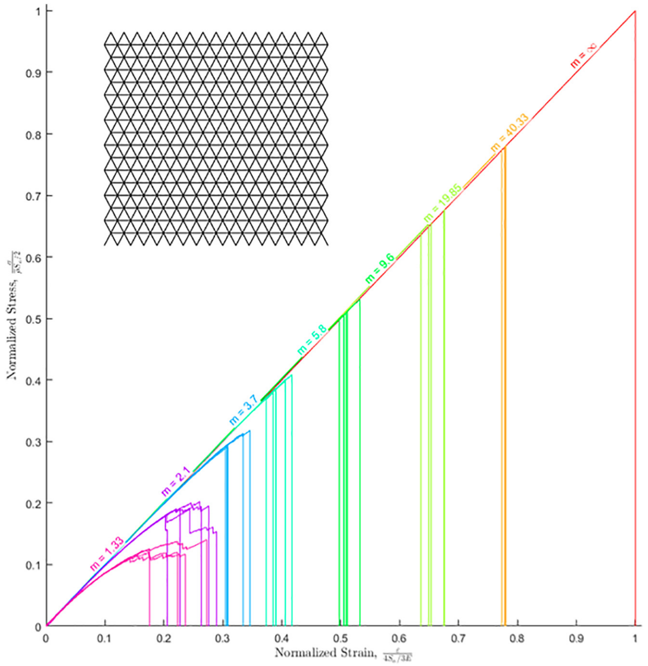


Figure 5. Stress–strain curves of samples without a crack. Each sample has 80 × 80 unit cells. Samples of various values of *m* are simulated. For each value of *m*, several randomizations of member strength are simulated.

For the ideal truss, the maximum displacement in the y-direction is $(bS_0/E)sin60° = 2bS_0/\sqrt{3}E$ for a single unit cell and is $2bS_0N_{u,y}/\sqrt{3}E$ for the entire truss. The maximum strain of the ideal truss is thus $4S_0/3E$. For a truss of stochastic member strength, we normalize the strain by the maximum strain of the ideal truss:

$$\frac{\varepsilon}{4S_0/3E} = \frac{\sqrt{3}E\delta}{2S_0bN_{u,y}}. \tag{6}$$

Simulations are performed on uncracked trusses. Runs are completed for several values of the shape factor $m > 1$. For each value for m, five runs are completed, each with a different randomization of strength. The simulated stress–strain curves are plotted for all runs (Figure 5). For large values of m (small scatter), the maximum stress tends to be high, and the stress–strain curves are nearly linear prior to the truss breaks. For small values of m (large scatter), the maximum stress is low, and the stress–strain curves are nonlinear.

We next turn to a truss with a crack of length c. We define the nominal stress by dividing the reaction force F by the ligament area:

$$\sigma = \frac{F}{\sqrt{A}b(N_{u,x} - c/b)}. \tag{7}$$

We nondimensionalize this nominal stress by the strength of the ideal truss without crack:

$$\frac{\sigma}{\rho S_0/2} = \frac{F}{\sqrt{3}AS_0(N_{u,x} - c/b)}. \tag{8}$$

Even for a truss in which all members have the same strength S_0, a crack concentrates stress at the crack tip, so the strength of such a truss needs to be calculated by a numerical simulation. Simulations were also completed for several values of the shape factor m. As an example, consider the stress–strain

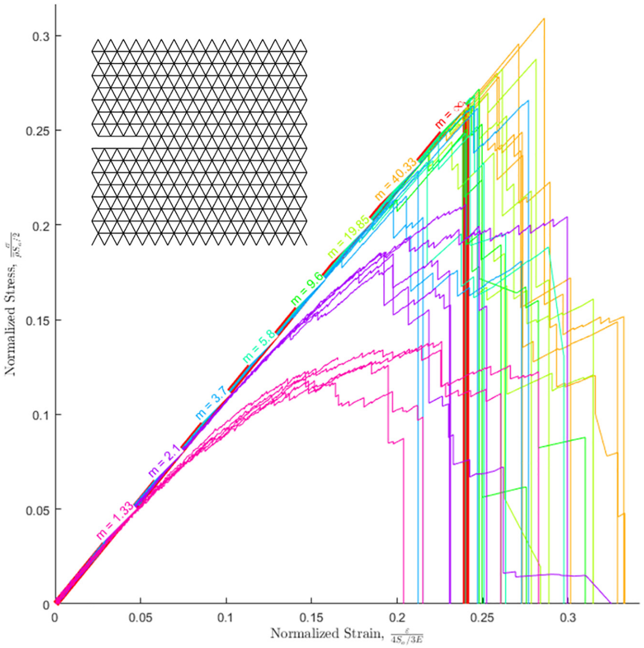


Figure 6. Stress–strain curves of samples with cracks, $c/b = 10$. Each sample has 80×80 unit cells. Samples of various values of m are simulated. For each value of m, trusses of several randomizations of member strength are simulated.

curves for a crack of length $c = 10b$. We compare trusses of the same crack length, but with various amounts of scatter (Figure 6). For a fixed m, the stress–strain curves vary from sample to sample.

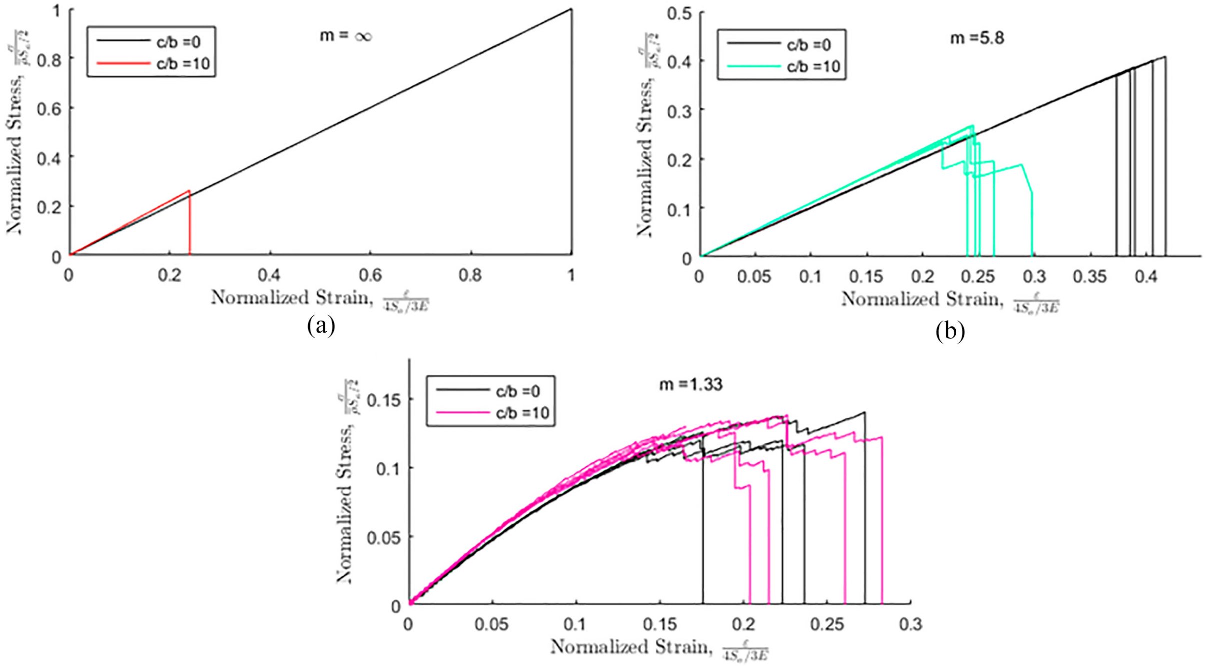
4. Flaw sensitivity

We next examine the competition between stress concentration and statistical distribution (Figure 7). As an example, consider the stress–strain curves for a crack of length $c = 10b$ (Figure 7). For the truss in which all members have the same strength, $m = \infty$, the stress–strain curve is linear until it reaches a peak then immediately decreases to zero. The first member to break is the member at the crack tip. After this member breaks, each subsequent member along the crack path will break without further increase of displacement. The crack tip concentrates stress, which knocks down the truss strength by a factor of 3.8 compared with the strength of a deterministic truss containing no crack (Figure 7(a)).

For a truss with a relatively small scatter in member strength, for example, $m = 5.8$ (Figure 7(b)), the truss with a crack has a lower strength than the truss without a crack. In addition, the stress concentration at the crack tip makes the truss sensitive to the strength of members in a small region around the crack. For $m = 5.8$, the scatter in member strength is not large relative to the stress concentration at the crack tip. Consequently, strong members at the crack tip can lead to a high truss strength and weak members at the crack tip can lead to a low truss strength.

Finally, consider trusses with large scatter in the member strength, for example, $m = 1.33$ (Figure 7(c)). The crack no longer knocks down the truss strength. The scatter in member strength is more significant than the stress concentration, so the presence of a crack does not knock down the truss strength. Here the crack can be arrested at a strong member, while away from the crack even without stress concentration, many members have low enough strength to break (Figure 2(b)).

The strength of the stochastic truss is understood in terms of the competition between stress concentration at the crack tip and statistical distribution of member strength. When member strength is deterministic, stress concentration at the crack tip prevails, and the truss fractures by breaking members at the crack tip, so that the truss is flaw-sensitive. When member strength has a large scatter, randomness prevails, and even though the stress is concentrated at the crack tip, the small volume around the crack tip has a low probability of having a member of low strength. By contrast, even though the stress is low



(a)

(b)

(c)

Figure 7. Effect of cracks on trusses of various amounts of scatter in member strength. (a) Deterministic truss, $m = \infty$. (b) Trusses of modest scatter in member strength, $m = 5.8$. (c) Trusses of large scatter in member strength, $m = 1.33$.

away from the crack tip, the large volume of the member corresponds to a high probability of having a low-strength member. When statistical distribution in member strength prevails, the truss is flaw-insensitive.

We plot truss strength as a function of crack length (Figure 8). Each curve in Figure 8 represents the average strength for trusses for one value of m, the X symbols represent the strength from each simulation. The black dotted line represents the scaling of strength with flaw size according to the Griffith theory, strength $\sim (c/b)^{-1/2}$. For a large m, there is good agreement between the scaling and the results of the simulations. The strength is highest for the deterministic truss and decreases as the scatter in the member strength is increased (m decreased). As the size of the crack is increased, the strength tends to decrease, indicating flaw sensitivity. However, there are regions where the curves are nearly horizontal, indicating flaw insensitivity. As the scatter in member strength increases. the trusses become insensitive to larger flaws.

5. Conclusion

We have studied the flaw sensitivity of stochastic trusses. Each truss is geometrically periodic, but the member strengths are assigned randomly according to the Weibull distribution. For each truss, with or without a crack, we simulate stress–strain curves using an algorithm in which members are progressively eliminated as they break. The strength of the truss is an outcome of a competition between the stress concentration at the crack tip and the statistical distribution of member strength. When the member strength scatters narrowly, the stress concentration prevails, and the strength of the truss is sensitive to flaws. When the member strength scatters widely, the scatter of strength over the large volume of the entire truss prevails, and the strength of the truss is insensitive to the flaws. The fractocohesive length scales with the length of each individual member, but the prefactor increases with the statistical variation of the members. When the member strength has wide statistical scatter, the fractocohesive length can be much larger than the length of the individual members.

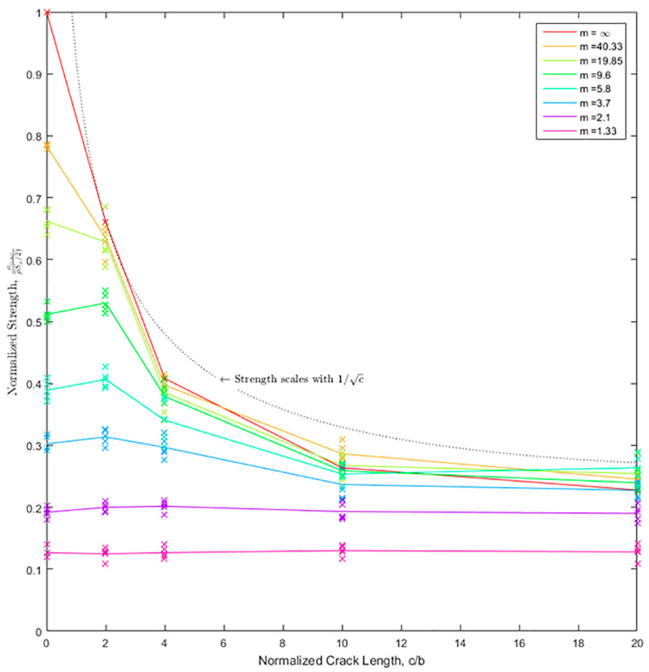


Figure 8. Truss strength as a function of crack length for samples with various values of *m*. For each value of *m*, trusses of several randomizations of member strength are simulated.

Declaration of conflicting interests

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